1 Transmission Link

Figure 1 shows a unified model for transmission lines and transformers, which we call a *transmission link*. A (directed) transmission link has a *from* end and a *to* end, and the currents and voltages at the two ends are related by

$$\begin{bmatrix} I_f \\ I_t \end{bmatrix} = \begin{bmatrix} n^2(y^s + y^m_{ft}) & -ne^{-j\theta^{\mathrm{sh}}}y^s \\ -ne^{j\theta^{\mathrm{sh}}}y^s & y^s + y^m_{tf} \end{bmatrix} \begin{bmatrix} V_f \\ V_t \end{bmatrix}.$$

It can be seen that the behavior of this transmission link is determined by the five parameters $\left(n, \theta^{\mathrm{sh}}, y^s, y^m_{ft}, y^m_{tf}\right)$.

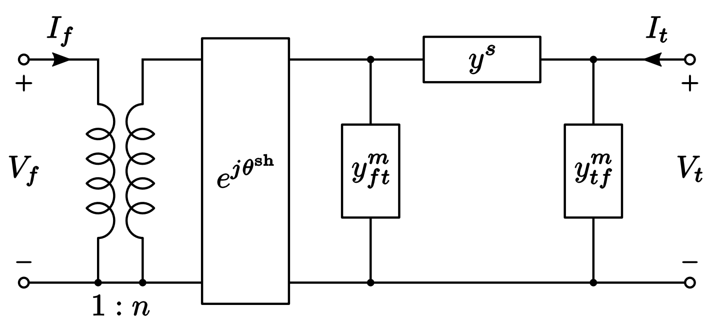

Figure 1: Circuit model for a transmission link.

For a normal power system, we have $n = 1$ and $\theta^{\mathrm{sh}} = 0$ for each transmission link.

2 Admittance Matrix

Suppose we have a power system with N buses which are indexed by $1, 2, \ldots, N$. For each transmission link connecting two buses, we pick one of the buses to be the *from* end and the other to be the *to* end. We assume that for each pair of buses there is at most one transmission link connecting the two buses. The set of transmission links is defined by

$$\mathcal{E} := \{(i,k) |\, \text{there is a transmission link with bus } i \text{ being the } \textit{from} \text{ bus and bus } k \text{ being the } \textit{to} \text{ bus}\}. \tag{1}$$

Note that if there is a transmission link connecting bus i and bus k, then either $(i,k) \in \mathcal{E}$ or $(k,i) \in \mathcal{E}$ (but not both), and which one is true depends on which of the two buses is assigned as the *from* bus of this transmission link. Throughout we always assume that the directed graph $G = (\{1,\ldots,N\},\mathcal{E})$ is weakly connected.

Each transmission link $(i,k) \in \mathcal{E}$ is associated with five parameters which we denote by

$$\left(n_{ik}, \theta_{ik}^{\mathrm{sh}}, y_{ik}^{s}, y_{ik}^{m}, y_{ki}^{m}\right)$$

as discussed in the previous section. Apart from the shunt admittance introduced by transmission links, we assume that for each bus i there can also be a linear load of admittance y_i connected to bus i; see Figure 2.

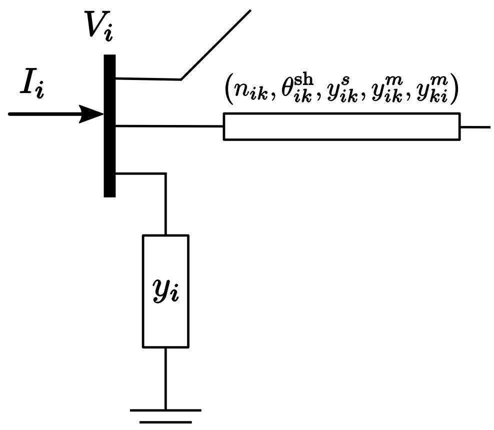

Figure 2: Diagram of bus i.

The admittance matrix $Y \in \mathbb{C}^{N\times N}$ is formed by the following steps:

Initialize: $Y \leftarrow \mathrm{diag}(y_1, y_2, \ldots, y_N)$, $S \leftarrow \mathcal{E}$

while $S \neq \varnothing$ **do**

Pick any $(i,k) \in S$, and remove (i,k) from S

$Y_{ii} \leftarrow Y_{ii} + n_{ik}^2\,(y_{ik}^s + y_{ik}^m)$

$Y_{kk} \leftarrow Y_{kk} + y_{ik}^s + y_{ki}^m$

$Y_{ik} \leftarrow -n_{ik} e^{-j\theta_{ik}^{\mathrm{sh}}} y_{ik}^s$

$Y_{ki} \leftarrow -n_{ik} e^{j\theta_{ik}^{\mathrm{sh}}} y_{ik}^s$

end while

Equivalently, we have

$$Y_{ii} = y_i + \sum_{k:(i,k)\in\mathcal{E}} n_{ik}^2 \left(y_{ik}^s + y_{ik}^m\right) + \sum_{k:(k,i)\in\mathcal{E}} \left(y_{ki}^s + y_{ik}^m\right),$$

$$Y_{ik} = \begin{cases} -\, n_{ik} e^{-j\theta_{ik}^{\rm sh}} y_{ik}^s, & \text{if } (i,k) \in \mathcal{E}, \\ -\, n_{ki} e^{j\theta_{ki}^{\rm sh}} y_{ki}^s, & \text{if } (k,i) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $I \in \mathbb{C}^N$ be the vector whose k'th entry is the total complex current injection at bus k, and let $V \in \mathbb{C}^N$ be the vector whose k'th entry is the complex voltage at bus k. Then we have

$$I = YV.$$

2.1 Kron Reduction

In a power system, the current injections are zero at buses where there are no loads or generators connected. Without loss of generality let's assume that the current injections at the last $N - M$ buses are zero. Then $I = YV$ can be written as

$$\begin{bmatrix} I_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix},$$

where

1. $I_1 \in \mathbb{C}^M$ is the vector of current injections at the first M buses,
2. $V_1 \in \mathbb{C}^M$ is the vector of voltages at the first M buses, $V_2 \in \mathbb{C}^{N-M}$ is the vector of voltages at the last $N - M$ buses,
3. $Y_{11} \in \mathbb{C}^{M\times M}, Y_{12} \in \mathbb{C}^{M\times(N-M)}, Y_{21} \in \mathbb{C}^{(N-M)\times M}, Y_{22} \in \mathbb{C}^{(N-M)\times(N-M)}$ are the blocks of the admittance matrix Y.

If Y_{22} is invertible, then $0 = Y_{21}V_1 + Y_{22}V_2$ implies $V_2 = -Y_{22}^{-1}Y_{21}V_1$, and so

$$I_1 = Y_{11}V_1 + Y_{12}V_2 = \left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right) V_1.$$

In this way we get a lower dimensional system in which the zero current injection buses are eliminated; the admittance matrix for this new system is $Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}$.

2.2 Solving for V When I is Known

Let's consider the situation where we know the current injections at each bus and want to find the complex voltages. Suppose Y is invertible, and then we can solve the linear equation

$$YV = I$$

to get V.

For large networks, solving the linear equation $YV = I$ can be time consuming. In many practical studies we need to solve $YV = I$ repeatedly with the same admittance matrix Y for the voltages corresponding to different sets of current injections. One way to reduce computation time in this situation is to employ LU factorization. First we apply LU factorization to Y to get

$$Y = LU$$

where L is lower-triangular with diagonal entries being 1 and U is upper-triangular. The LU factorization has time-complexity $O(N^3)$. Then we solve the following two equations:

$$L\tilde{V} = I, \qquad UV = \tilde{V}.$$

The first equation can be solved by forward substitution because L is lower-triangular, and the second equation can be solved by backward substitution because U is upper-triangular. The time complexity of solving these two equations is $O(N^2)$.

The details of LU factorization can be found in Steven's lecture notes, or text-books on numerical algorithms (e.g. see Section 28.1 of *Introduction to Algorithms*, 3rd ed. by T. H. Cormen, et al.). In MATLAB, *LU* factorization can be computed by the `lu` function.

3 Power Flow Analysis

3.1 Power Flow Equations

Suppose we have a power system of N buses indexed by $1, 2, \ldots, N$, and the admittance matrix is Y.

Let $V \in \mathbb{C}^N$ be the vector of complex voltages, $I \in \mathbb{C}^N$ be the vector of current injections, and $s = p + jq \in \mathbb{C}^N$ be the vector of complex power injections. Then

$$s_i = V_i I_i^* = V_i \left(\sum_{k=1}^{N} Y_{ik} V_k \right)^* = V_i \sum_{k=1}^{N} Y_{ik}^* V_k^*, \qquad i = 1, \ldots, N.$$

These equations are called the *power flow equations*.

Polar form Let θ_i denote the phase angle of the complex voltage V_i. Then

$$\begin{aligned} s_i &= \sum_{k=1}^{N} |V_i||V_k| e^{j(\theta_i - \theta_k)} Y_{ik}^* = \sum_{k=1}^{N} |V_i||V_k| \left(\cos(\theta_i - \theta_k) + j \sin(\theta_i - \theta_k) \right) \left(G_{ik} - jB_{ik} \right) \\ &= \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right) \\ &\qquad + j \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right), \end{aligned}$$

and so

$$\begin{aligned} p_i &= \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right) \\ &= G_{ii} |V_i|^2 + \sum_{k \neq i} |V_i||V_k| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right), \end{aligned} \tag{2a}$$

$$\begin{aligned} q_i &= \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right) \\ &= -B_{ii} |V_i|^2 + \sum_{k \neq i} |V_i||V_k| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right), \end{aligned} \tag{2b}$$

where $G_{ik} = \mathrm{Re}\{Y_{ik}\}$, $B_{ik} = \mathrm{Im}\{Y_{ik}\}$, p_i is the real power injection and q_i is the reactive power injection at bus i.

Cartesian form We can also perform the computation in the Cartesian form $V_i = V_{i,\mathrm{R}} + jV_{i,\mathrm{I}}$. Then it can be shown that

$$\begin{aligned}
p_i &= \sum_{k=1}^{N} \left(G_{ik}\left(V_{i,\mathrm{R}}V_{k,\mathrm{R}} + V_{i,\mathrm{I}}V_{k,\mathrm{I}}\right) + B_{ik}\left(V_{i,\mathrm{I}}V_{k,\mathrm{R}} - V_{i,\mathrm{R}}V_{k,\mathrm{I}}\right)\right) \\
&= G_{ii}\left(V_{i,\mathrm{R}}^2 + V_{i,\mathrm{I}}^2\right) + \sum_{k\neq i} \left(G_{ik}\left(V_{i,\mathrm{R}}V_{k,\mathrm{R}} + V_{i,\mathrm{I}}V_{k,\mathrm{I}}\right) + B_{ik}\left(V_{i,\mathrm{I}}V_{k,\mathrm{R}} - V_{i,\mathrm{R}}V_{k,\mathrm{I}}\right)\right), \\
q_i &= \sum_{k=1}^{N} \left(G_{ik}\left(V_{i,\mathrm{I}}V_{k,\mathrm{R}} - V_{i,\mathrm{R}}V_{k,\mathrm{I}}\right) - B_{ik}\left(V_{i,\mathrm{R}}V_{k,\mathrm{R}} + V_{i,\mathrm{I}}V_{k,\mathrm{I}}\right)\right) \\
&= -B_{ii}\left(V_{i,\mathrm{R}}^2 + V_{i,\mathrm{I}}^2\right) + \sum_{k\neq i} \left(G_{ik}\left(V_{i,\mathrm{I}}V_{k,\mathrm{R}} - V_{i,\mathrm{R}}V_{k,\mathrm{I}}\right) - B_{ik}\left(V_{i,\mathrm{R}}V_{k,\mathrm{R}} + V_{i,\mathrm{I}}V_{k,\mathrm{I}}\right)\right).
\end{aligned}$$

They can also be written in the compact form

$$\begin{aligned}
p_i &= \frac{1}{2}\begin{bmatrix} V_\mathrm{R} \\ V_\mathrm{I} \end{bmatrix}^T \begin{bmatrix} G_i + G_i^T & -B_i + B_i^T \\ B_i - B_i^T & G_i + G_i^T \end{bmatrix} \begin{bmatrix} V_\mathrm{R} \\ V_\mathrm{I} \end{bmatrix}, \\
q_i &= \frac{1}{2}\begin{bmatrix} V_\mathrm{R} \\ V_\mathrm{I} \end{bmatrix}^T \begin{bmatrix} -B_i - B_i^T & -G_i + G_i^T \\ G_i - G_i^T & -B_i - B_i^T \end{bmatrix} \begin{bmatrix} V_\mathrm{R} \\ V_\mathrm{I} \end{bmatrix}
\end{aligned}$$

where $V_\mathrm{R} = \mathrm{Re}\{V\}$, $V_\mathrm{I} = \mathrm{Im}\{V\}$, $G_i = \mathrm{Re}\{Y_i\}$, $B_i = \mathrm{Im}\{Y_i\}$, and Y_i is an $N \times N$ complex matrix whose i'th row is equal to the i'th row of Y and whose other entries are zero.

3.2 The Power Flow Problem

Roughly speaking, by solving "the power flow problem", we mean finding the complex voltages for all the buses of the power system when the complex power injections are given such that the power injections and the voltages satisfy the power flow equations. However, there are several considerations:

1. In general we cannot specify all the complex power injections independently, as can be seen from the following two examples:

   (a) Consider the case where all the transmission lines are lossless (i.e. y^s and y^m are purely imaginary). Then by the conservation of power,

   $$\sum_{i=1}^{N} p_i = 0,$$

   which puts a constraint on the power injection vector.

(b) Consider the case where the power system consists of two buses connected by a short transmission line with series admittance $y = g + jb$. Then the admittance matrix is

$$Y = \begin{bmatrix} g + jb & -g - jb \\ -g - jb & g + jb \end{bmatrix},$$

and the power flow equations are

$$\begin{aligned} p_1 &= g|V_1|^2 + |V_1||V_2|(-g\cos\theta_2 + b\sin\theta_2), \\ p_2 &= g|V_2|^2 + |V_1||V_2|(-g\cos\theta_2 - b\sin\theta_2), \\ q_1 &= -b|V_1|^2 + |V_1||V_2|(g\sin\theta_2 + b\cos\theta_2), \\ q_2 &= -b|V_2|^2 + |V_1||V_2|(-g\sin\theta_2 + b\cos\theta_2), \end{aligned}$$

and so

$$\begin{aligned} p_1 + p_2 &= g(|V_1|^2 + |V_2|^2 - 2|V_1||V_2|\cos\theta_2) = g|V_1 - V_2|^2, \\ q_1 + q_2 &= -b(|V_1|^2 + |V_2|^2 - 2b|V_1||V_2|\cos\theta_2) = -b|V_1 - V_2|^2, \end{aligned}$$

(these two equalities can also be obtained by the conservation of complex power), which imply that

$$b(p_1 + p_2) + g(q_1 + q_2) = 0.$$

These two examples suggest that there is a constraint imposed on the power injections by the need to balance complex power in steady-state operation.

2. The solution in general is not unique: If the complex voltages $V \in \mathbb{C}^N$ and the complex power injections $s \in \mathbb{C}^N$ satisfy the power flow equations, then $e^{j\theta_0}V$ and s will also satisfy the power flow equations for any $\theta_0 \in \mathbb{R}$. This means that we need to fix the voltage phase angle at a certain bus.

3. For transmission networks, it is reasonable to model loads as constant power loads, meaning that the complex power consumed by the loads are not affected by the voltages imposed on the loads. This is partly related to the fact that in transmission networks, loads are usually connected to the network via substations, in which there are voltage regulators (transformers with variable voltage gains) to keep the voltages on the secondary side roughly constant regardless of the voltages on the primary side.

   On the other hand, for buses with bulk generators connected it is more reasonable to specify their real power injections and voltage magnitudes (this is due to how the bulk generators are operated, which we will not talk about in detail in this course); the reactive power injections will then be determined after the complex voltages are solved.

These considerations lead to the following formulation of the power flow problem that is employed by power system researchers and engineers:

- There are three types of buses:

  1. A *slack bus* or *swing bus*, at which the voltage phase angle is zero and the voltage magnitude is given. In other words, $V_1 = |V_1|$ is specified.

     The slack bus is placed at a generator bus and is usually bus 1.

  2. *PV buses* or *voltage-controlled buses*, at which the real power injections and the voltage magnitudes are specified. These buses are usually buses with bulk generators connected.

     Suppose bus i is a PV bus connected to generators injecting real power p_{Gi} and loads consuming real power P_{Di}, then the net real power injection at bus i is given by $p_i = p_{Gi} - p_{Di}$.

     Without loss of generality we assume that the PV buses are bus 2 to bus M.

  3. *PQ buses* or *load buses*, at which the real and reactive power injections are specified. These buses are usually the load buses.

     Suppose bus i is a PQ bus connected to loads consuming real power p_{Di} and reactive power q_{Di}, then the net real and reactive complex power injections at bus i are given by $p_i = -p_{Di}$, $q_i = -q_{Di}$.

- The power flow problem is to find the following $2N - M - 1$ real quantities

$$\theta_2, \dots, \theta_M, |V_{M+1}|, \theta_{M+1}, |V_{M+2}|, \theta_{M+2}, \dots, |V_N|, \theta_N \tag{3}$$

  from the following $2N - M - 1$ power flow equations

$$p_i = \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k)\right), \ i = 2, \dots, N, \tag{4a}$$

$$q_i = \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik}\sin(\theta_i - \theta_k) - B_{ik}\cos(\theta_i - \theta_k)\right), \ i = M+1, \dots, N. \tag{4b}$$

  given the following specified quantities

$$|V_1|, \theta_1 = 0, p_2, |V_2|, \dots, p_M, |V_M|, p_{M+1}, q_{M+1}, \dots p_N, q_N. \tag{5}$$

- After the $2N - M - 1$ unknown quantities in (3) are found, the complex power injection at the slack bus and the reactive power injections at the PV buses are calculated by

$$p_1 = \sum_{k=1}^{N} |V_1||V_k| \left(G_{1k} \cos\theta_k - B_{1k} \sin\theta_k\right),$$

$$q_i = \sum_{k=1}^{N} |V_i||V_k| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k)\right), \ i = 1, \ldots, M.$$

  The real and reactive power generated by generators connected to the slack bus are $p_{G1} = p_1 + p_{D1}$ and $q_{G1} = q_1 + q_{D1}$, where p_{D1} and q_{D1} are the real and reactive power consumption of the load connected to the slack bus. For PV bus i, The reactive power generated by generators connected to it is $q_{Gi} = q_i + q_{Di}$, where q_{Di} is the reactive power consumption of the load connected to bus i.

- Moreover, after all the voltages are found, we can then calculate the complex powers and currents that flow through the transmission links, to check if operational constraints (e.g. thermal limits) are satisfied.

Remark. 1. The slack bus plays two roles: (1) to serve as a reference of the voltage phase angles, (2) to achieve power balance of the whole system by not specifying its power injections. Then, since the complex power injection of the slack bus is not predetermined, we will need generators connected to the slack bus so that its complex power injection is adjustable.

2. For a PV bus i, usually the reactive power produced by the generators q_{Gi} must be constrained within a certain range such as $q_{Gi}^{\min} \leq q_{Gi} \leq q_{Gi}^{\max}$. After solving the power flow problem and evaluating the resulting reactive power generation q_{Gi}, if it exceeds one of the limits then q_{Gi} will be set to that limit and bus i will be re-classified as a PQ bus with $|V_i|$ to be determined. The updated power flow equations are then re-solved for the unknown quantities.

The core of solving the power flow problem is solving the $2N - M - 1$ nonlinear power flow equations. These nonlinear equations may have no, unique, or multiple solutions. The following subsections introduce numerical methods for solving the nonlinear power flow equations.

3.3 The Newton-Raphson Method

We first give a concise description of the general Newton-Raphson algorithm.

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function whose entries are all continuously differentiable. We want to find the solution to the following equation

$$f(x) = 0.$$

The basic idea of the Newton-Raphson method is as follows: Suppose $x^k \in \mathbb{R}^n$ is some point sufficiently close to the true solution. We approximate $f(x)$ by its first-order Taylor expansion at x^t

$$f(x^k) + J_f(x^k)(x - x^k),$$

where $J_f(x^k)$ is the Jacobian of f evaluated at x_k:

$$J_f(x^k) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^k) & \frac{\partial f_1}{\partial x_2}(x^k) & \cdots & \frac{\partial f_1}{\partial x_n}(x^k) \\ \frac{\partial f_2}{\partial x_1}(x^k) & \frac{\partial f_2}{\partial x_2}(x^k) & \cdots & \frac{\partial f_2}{\partial x_n}(x^k) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^k) & \frac{\partial f_n}{\partial x_2}(x^k) & \cdots & \frac{\partial f_n}{\partial x_n}(x^k) \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

We then solve the linear approximation of $f(x) = 0$, i.e., the equation that the first-order Taylor expansion of f at x^k is equal to zero. Its solution will be denoted by x^{k+1} and satisfies

$$J_f(x^k)(x^{k+1} - x^k) = -f(x^k), \tag{6}$$

We then use x^{k+1} as the next iterate and repeat the above process until the iterates converge to a point x^*. We now get a solution x^* that satisfies $f(x^*) = 0$.

The Newton-Raphson method is widely used in numerically solving nonlinear equations. It may produce a diverging sequence of iterates if the initial iterate x^0 is not sufficiently close to a solution, or if $J_f(x)$ is not well-behaved around a solution. There is also no guarantee that all the solutions can be found. However, under certain conditions, it can be shown that the Newton-Raphson method converges at a quadratic rate. In practice, it is often observed that only a few iterations are needed by the Newton-Raphson method to produce a very accurate solution. See textbooks on numerical algorithms for more details.

Now we apply the Newton-Raphson method on solving the power flow problem.

We use the polar form of the power flow equations. Denote

$$\underbrace{\theta = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}, \quad |V| = \begin{bmatrix} |V_{M+1}| \\ \vdots \\ |V_N| \end{bmatrix}}_{\text{unknown variables to be found}}, \qquad \underbrace{p = \begin{bmatrix} p_2 \\ \vdots \\ p_N \end{bmatrix}, \quad q = \begin{bmatrix} q_{M+1} \\ \vdots \\ q_N \end{bmatrix}}_{\text{specified quantities}},$$

and define the following functions

$$\begin{aligned} \mathsf{p}_i(\theta, |V|) &= G_{ii}|V_i|^2 + \sum_{k \neq i} |V_i||V_k| \left(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right), \\ \mathsf{q}_i(\theta, |V|) &= -B_{ii}|V_i|^2 + \sum_{k \neq i} |V_i||V_k| \left(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right), \end{aligned} \tag{7}$$

$$\mathsf{p}(\theta, |V|) = \begin{bmatrix} \mathsf{p}_2(\theta, |V|) \\ \vdots \\ \mathsf{p}_N(\theta, |V|) \end{bmatrix}, \quad \mathsf{q}(\theta, |V|) = \begin{bmatrix} \mathsf{q}_{M+1}(\theta, |V|) \\ \vdots \\ \mathsf{q}_N(\theta, |V|) \end{bmatrix}.$$

The $2N - M - 1$ power flow equations can then be written as

$$\begin{bmatrix} \mathsf{p}(\theta, |V|) - p \\ \mathsf{q}(\theta, |V|) - q \end{bmatrix} = 0.$$

The Newton-Raphson iterations for solving the power flow problem are then given by the following equations

$$\begin{bmatrix} J_{\mathsf{p},\theta}(\theta^k, |V|^k) & J_{\mathsf{p},|V|}(\theta^k, |V|^k) \\ J_{\mathsf{q},\theta}(\theta^k, |V|^k) & J_{\mathsf{q},|V|}(\theta^k, |V|^k) \end{bmatrix} \begin{bmatrix} \Delta\theta^k \\ \Delta|V|^k \end{bmatrix} = \begin{bmatrix} p - \mathsf{p}(\theta^k, |V|^k) \\ q - \mathsf{q}(\theta^k, |V|^k) \end{bmatrix}, \tag{8a}$$

$$\begin{bmatrix} \theta^{k+1} \\ |V|^{k+1} \end{bmatrix} = \begin{bmatrix} \theta^k \\ |V|^k \end{bmatrix} + \begin{bmatrix} \Delta\theta^k \\ \Delta|V|^k \end{bmatrix}, \tag{8b}$$

where

$$J_{\mathsf{p},\theta}(\theta, |V|) := \begin{bmatrix} \frac{\partial \mathsf{p}_2}{\partial \theta_2}(\theta, |V|) & \cdots & \frac{\partial \mathsf{p}_2}{\partial \theta_N}(\theta, |V|) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathsf{p}_N}{\partial \theta_2}(\theta, |V|) & \cdots & \frac{\partial \mathsf{p}_N}{\partial \theta_N}(\theta, |V|) \end{bmatrix}, \tag{9a}$$

$$J_{\mathsf{q},\theta}(\theta,|V|) := \begin{bmatrix} \frac{\partial \mathsf{q}_{M+1}}{\partial \theta_2}(\theta,|V|) & \cdots & \frac{\partial \mathsf{q}_{M+1}}{\partial \theta_N}(\theta,|V|) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathsf{q}_{N}}{\partial \theta_2}(\theta,|V|) & \cdots & \frac{\partial \mathsf{q}_{N}}{\partial \theta_N}(\theta,|V|) \end{bmatrix}, \tag{9b}$$

$$J_{\mathsf{p},|V|}(\theta,|V|) := \begin{bmatrix} \frac{\partial \mathsf{p}_{2}}{\partial |V_{M+1}|}(\theta,|V|) & \cdots & \frac{\partial \mathsf{p}_{2}}{\partial |V_N|}(\theta,|V|) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathsf{p}_{N}}{\partial |V_{M+1}|}(\theta,|V|) & \cdots & \frac{\partial \mathsf{p}_{N}}{\partial |V_N|}(\theta,|V|) \end{bmatrix}, \tag{9c}$$

$$J_{\mathsf{q},|V|}(\theta,|V|) := \begin{bmatrix} \frac{\partial \mathsf{q}_{M+1}}{\partial |V_{M+1}|}(\theta,|V|) & \cdots & \frac{\partial \mathsf{q}_{M+1}}{\partial |V_N|}(\theta,|V|) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathsf{q}_{N}}{\partial |V_{M+1}|}(\theta,|V|) & \cdots & \frac{\partial \mathsf{q}_{N}}{\partial |V_N|}(\theta,|V|) \end{bmatrix}, \tag{9d}$$

and

$$\frac{\partial \mathsf{p}_i}{\partial \theta_k}(\theta,|V|) = \begin{cases} |V_i||V_k|(G_{ik}\sin(\theta_i-\theta_k) - B_{ik}\cos(\theta_i-\theta_k)), & i \neq k, \\ -\mathsf{q}_i(\theta,|V|) - B_{ii}|V_i|^2, & i = k, \end{cases} \tag{10a}$$

$$\frac{\partial \mathsf{p}_i}{\partial |V_k|}(\theta,|V|) = \begin{cases} |V_i|(G_{ik}\cos(\theta_i-\theta_k) + B_{ik}\sin(\theta_i-\theta_k)), & i \neq k, \\ \frac{\mathsf{p}_i(\theta,|V|)}{|V_i|} + G_{ii}|V_i|, & i = k, \end{cases} \tag{10b}$$

$$\frac{\partial \mathsf{q}_i}{\partial \theta_k}(\theta,|V|) = \begin{cases} -|V_i||V_k|(G_{ik}\cos(\theta_i-\theta_k) + B_{ik}\sin(\theta_i-\theta_k)), & i \neq k, \\ \mathsf{p}_i(\theta,|V|) - G_{ii}|V_i|^2, & i = k, \end{cases} \tag{10c}$$

$$\frac{\partial \mathsf{q}_i}{\partial |V_k|}(\theta,|V|) = \begin{cases} |V_i|(G_{ik}\sin(\theta_i-\theta_k) - B_{ik}\cos(\theta_i-\theta_k)), & i \neq k, \\ \frac{\mathsf{q}_i(\theta,|V|)}{|V_i|} - B_{ii}|V_i|, & i = k, \end{cases} \tag{10d}$$

The textbook also introduces a modified version of the Newton-Raphson iterations that utilize the relationships between the elements of the Jacobian submatrices to reduce data storage burden.

3.4 Decoupled Power Flow Method

In a well-designed and properly operated transmission network, we usually have the following facts:

1. The transmission links are mostly reactive, meaning that

$$G_{ik} \ll B_{ik}$$

   for any bus i and bus k that are connected by a transmission link.

2. The phase angle difference $\theta_i - \theta_k$ is very small for any bus i and bus k that are connected by a transmission link. Therefore

$$\sin(\theta_i - \theta_k) \ll \cos(\theta_i - \theta_k).$$

By comparing these observations with (10) and (7), we can see that, for $(\theta, |V|)$ that is equal or sufficiently close to a real operating point of the power system, the entries of $J_{\mathsf{p},|V|}(\theta, |V|)$ and $J_{\mathsf{q},\theta}(\theta, |V|)$ are very small compared to the entries of $J_{\mathsf{p},\theta}(\theta, |V|)$ and $J_{\mathsf{q},|V|}(\theta, |V|)$, which suggests that the Jacobian can be approximated by

$$\begin{bmatrix} J_{\mathsf{p},\theta}(\theta, |V|) & 0 \\ 0 & J_{\mathsf{q},|V|}(\theta, |V|) \end{bmatrix}.$$

This means that the voltage magnitudes and the real power injections are approximately decoupled, and the voltage angles and the reactive power injections are approximately decoupled. By applying this approximation in the Newton-Raphson iterations, we get the decoupled power flow method given by

$$\begin{aligned} J_{\mathsf{p},\theta}(\theta^k, |V|^k)\Delta\theta^k &= p - \mathsf{p}(\theta^k, |V|^k) \\ J_{\mathsf{q},|V|}(\theta^k, |V|^k)\Delta|V|^k &= q - \mathsf{q}(\theta^k, |V|^k) \end{aligned} \tag{11a}$$

$$\begin{bmatrix} \theta^{k+1} \\ |V|^{k+1} \end{bmatrix} = \begin{bmatrix} \theta^k \\ |V|^k \end{bmatrix} + \begin{bmatrix} \Delta\theta^k \\ \Delta|V|^k \end{bmatrix}, \tag{11b}$$

The decoupled power flow method can be further simplified with more approximations; see the course's textbook. These approximations and simplifications help to reduce the computation time of solving power flow equations.

3.5 DC Power Flow

The DC power flow model is a linear power flow model that makes the following assumptions

1. $G_{ii} \approx 0$ and $G_{ik} \approx 0$ for all transmission links. In other words, the power system is *lossless*.

2. The phase angle difference $\theta_i - \theta_k$ is small so that $\sin(\theta_i - \theta_k) \approx \theta_i - \theta_k$ for any bus i and bus k connected by a transmission link.

3. $|V_i| \approx 1$ for all i, i.e., the voltage magnitudes are near their nominal values at all buses.

4. Reactive power injections can be ignored.

The last two assumptions follow from the decoupling of p and $|V|$ and the decoupling of q and θ, which suggest that reactive power injections can be chosen to stabilize the voltage magnitudes separately from θ and p. By applying these approximations to the original power flow equations, we get

$$p_i = \sum_{k=1}^{N} B_{ik}(\theta_i - \theta_k). \tag{12}$$

Now let's consider the case of a normal power system. Let $\mathcal{E}$ be the set of (directed) transmission links [see Eq. (1)], and E be the number of elements in $\mathcal{E}$. We assign an arbitrary order to the set of transmission links so that we can talk about the k'th transmission link for $k = 1, \ldots, E$. Let the incidence matrix $M \in \mathbb{R}^{N \times E}$ be defined by

$$M_{ik} = \begin{cases} 1, & \text{bus } i \text{ is the } \textit{from} \text{ bus of the } k\text{'th transmission link,} \\ -1, & \text{bus } i \text{ is the } \textit{to} \text{ bus of the } k\text{'th transmission link,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $X \in \mathbb{R}^{E \times E}$ be the diagonal matrix whose k'th diagonal is equal to the series reactance of the k'th transmission line. Let $p \in \mathbb{R}^N$ and $\theta \in \mathbb{R}^N$ denote the vector of real power injections and voltage phase angles of all the N buses. Then it can be shown that (12) can be written as

$$p = MX^{-1}M^T\theta.$$

Furthermore, it can be verified that the k'th entry of the vector

$$P = X^{-1}M^T\theta$$

is equal to the real power flow through the k'th transmission link.